

ON FREE PRODUCTS OF POLISH GROUPS

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ABSTRACT. Using generalized Graev metrics we introduce a notion of free products of Polish groups and study its basic properties. We also show that any two Polish groups G and H can be embedded into a larger Polish group in such a way that the subgroup generated by G and H is naturally isomorphic as an abstract group to the free product $G * H$.

1. INTRODUCTION

In recent years L. Ding and S. Gao significantly expanded the theory of Graev metrics and introduced generalized Graev metrics, which turned out to be very useful in approaching questions about existence of surjectively universal objects (see [2–4]). In the category of abstract groups free products can be characterized by their universal property: every homomorphism from factors extends uniquely to a homomorphism from the free product. Based on the point of view of generalized Graev groups we introduce free products of Polish groups and describe the universal property it satisfies. In Theorem 3.5 we also show that for Polish groups G and H one can find a Polish group T and embeddings $\psi_G : G \rightarrow T$, $\psi_H : H \rightarrow T$ such that the natural homomorphism (of abstract groups) $\psi : G * H \rightarrow \langle \psi_G(G), \psi_H(H) \rangle$ is an isomorphism.

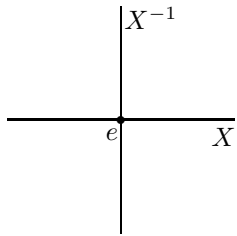
To fix terminology recall that a *Polish group* is a separable completely metrizable topological group. A *metric group* is a pair (G, d) , where G is a topological group and d is a compatible left-invariant metric. We say that a map $\phi : X \rightarrow Y$ between metric spaces (X, d_X) and (Y, d_Y) is *Lipschitz* if $d_Y(\phi(x_1), \phi(x_2)) \leq d_X(x_1, x_2)$ for all $x_1, x_2 \in X$. Such maps are usually called 1-Lipschitz, but since we shall not use K -Lipschitz maps for any other K , the constant will be omitted. A homomorphism $\phi : G \rightarrow H$ between topological groups is called an *embedding* if it is continuous, injective, and its inverse $\phi^{-1} : \phi(G) \rightarrow G$ is also continuous.

Recall that any metrizable topological group G admits a group completion: if d is a compatible left-invariant metric on G , then $d'(g_1, g_2) = d(g_1^{-1}, g_2^{-1})$ is a compatible right-invariant metric and the Hausdorff completion of the metric space $(G, d + d')$ admits a unique extension of group operations, which turns the completion into a completely metrizable group. If G is a metrizable topological group, G_1 is the completion of G and $\phi : G \rightarrow H$ is a continuous homomorphism into a completely metrizable group H , then ϕ extends to a continuous homomorphism $\phi : G_1 \rightarrow H$ (see, for example, [1, p. 6]).

A *norm* on a group G is a function $N : G \rightarrow \mathbb{R}^+$ such that

- (i) $N(f) = 0$ if and only if $f = e$;
- (ii) $N(f) = N(f^{-1})$;
- (iii) $N(f_1 f_2) \leq N(f_1) + N(f_2)$.

Any norm gives rise to a left-invariant metric $d(f_1, f_2) = N(f_1^{-1} f_2)$, and for any left-invariant metric d the function $f \mapsto d(f, e)$ is a norm.



We now recall the framework and some results from Ding–Gao [4]. A *pointed metric space* is a triple (X, e, d) , where (X, d) is a metric space and $e \in X$ is a distinguished point. The symbol e will denote distinguished points of pointed metric spaces and identity elements of groups; to which group or metric space it is referred will be clear from the context. Let X^{-1} denote a copy of X with elements of X^{-1} being formal inverses of the elements of X with the agreement $X \cap X^{-1} = \{e\}$, that is, $e^{-1} = e$. Extend d to a metric on X^{-1} by declaring $d(x^{-1}, y^{-1}) = d(x, y)$ for all $x, y \in X$. Let (\bar{X}, e, d) denote the amalgam of (X, d) and (X^{-1}, d) over the subspace $e \in X \cap X^{-1}$. In other words, $\bar{X} = X \cup X^{-1}$ and $d(x, y^{-1}) = d(x, e) + d(e, y)$. We can extend the inverse $x \mapsto x^{-1}$ to a function on \bar{X} by setting $(x^{-1})^{-1} = x$. To summarize, starting from a pointed metric

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space (X, e, d) we constructed in a canonical way a pointed metric space (\overline{X}, e, d) , X is a subspace of \overline{X} and the function $\overline{X} \ni x \mapsto x^{-1}$ is an isometric involution. We shall say that \overline{X} is obtained from X by *adding formal inverses*.

Definition 1.1 (Ding–Gao). A *scale* on a pointed metric space (X, e, d) is a function $\Gamma : X \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying for all $x \in X$ and all $r \in \mathbb{R}^+$

- (i) $\Gamma(e, r) = r$, $\Gamma(x, r) \geq r$;
- (ii) $\Gamma(x, r) = 0$ if and only if $r = 0$;
- (iii) $\Gamma(x, \cdot)$ is a monotone increasing function with respect to the second variable;
- (iv) $\lim_{r \rightarrow 0} \Gamma(x, r) = 0$.

By a *scaled space* we mean a tuple $(\overline{X}, e, d, \Gamma)$, where \overline{X} is obtained from some pointed metric space X by adding formal inverses and Γ is a scale on \overline{X} . We shall denote scaled spaces with bold letters \mathbf{X} , \mathbf{Y} , etc.

For a set X let $W(X)$ denote the set of nonempty words in the alphabet X . Let $w \in W(X)$ be a word of the form $w = x_1 \cdots x_n$ for some $x_i \in X$. Integer n is the length of the word w and it is denoted by $|w|$. A *match* on w is a bijection $\theta : \{x_1, \dots, x_n\} \rightarrow \{x_1, \dots, x_n\}$ such that $\theta(\theta(x_i)) = x_i$ for all i , and there are no $i < j$ such that $\theta(x_i) = x_k$, $\theta(x_j) = x_l$ and $i < j < k < l$. In other words we can think of a match as a set of arcs connecting letters of w such that two arcs are either disjoint, or one of the arcs is contained in the other arc. A *subword* of w is any word u of the form $x_{i_1} \cdots x_{i_k}$ for some $1 \leq i_1 < i_2 < \dots < i_k \leq n$. If θ is a match on w and $\theta(\{x_{i_1}, \dots, x_{i_k}\}) = \{x_{i_1}, \dots, x_{i_k}\}$, then the *restriction* of θ on the set $\{x_{i_1}, \dots, x_{i_k}\}$ is a match on u , but we shall abuse terminology and say that θ itself is a match on u in this case. The concatenation of words w_1 and w_2 is denoted by $w_1 w_2$.

Let $\mathbf{X} = (\overline{X}, e, d, \Gamma)$ be a scaled space. We stress that $W(\overline{X})$ denotes the set of *all* words, as opposed to denoting the set of reduced words. Following [4], for a match θ on a word $w \in W(\overline{X})$ we define a number $N^\theta(w)$ by induction on the length of w as follows.

- (i) If $w = x$ for some $x \in \overline{X}$, then $N^\theta(w) = d(x, e)$; if $w = x_1 x_2$ and $\theta(x_1) = x_2$, then $N^\theta(w) = d(x_1, x_2^{-1})$.
- (ii) If $\theta(x_1) = x_k$ and $k < |w|$, then $w = u_1 u_2$ for some words $u_i \in W(\overline{X})$ with $|u_1| = k$, $|u_2| = n - k$, θ is a match on both u_1 and u_2 , and we set $N^\theta(w) = N^\theta(u_1) + N^\theta(u_2)$.
- (iii) If $\theta(x_1) = x_n$, $n = |w|$, then let $w = x_1 u x_n$ for $u \in W(\overline{X})$, $x_1, x_n \in \overline{X}$, θ is a match on u and we set

$$N^\theta(w) = d(x_1, x_n^{-1}) + \min \left\{ \Gamma(x_1^{-1}, N^\theta(u)), \Gamma(x_n, N^\theta(u)) \right\}.$$

Let $F(X \setminus \{e\})$ be the free group over the set $X \setminus \{e\}$. We view \overline{X} as a subset of $F(X \setminus \{e\})$ and $e \in X$ is identified with the identity element of $F(X \setminus \{e\})$. There is a natural evaluation map $\hat{\cdot} : W(\overline{X}) \rightarrow F(X \setminus \{e\})$ that sends a word to the product of its letters; this map is surjective. The norm $N(f)$ of $f \in F(X \setminus \{e\})$ is defined by

$$N(f) = \inf \left\{ N^\theta(w) : w \in W(\overline{X}), \hat{w} = f \text{ and } \theta \text{ is a match on } w \right\}.$$

Proposition 1.2 (Lemma 3.6, Theorem 3.9, [4]). *The function $f \mapsto N(f)$ is a norm on the group $F(X \setminus \{e\})$, and the latter is a topological group in the topology of N . The natural inclusion map $\overline{X} \hookrightarrow F(X \setminus \{e\})$ is an isometry.*

The norm N is called *the Graev norm of the scale Γ* . We shall denote by $F(\mathbf{X})$ the free group $F(X \setminus \{e\})$ together with the Graev norm N and we shall view \overline{X} as a subset of $F(\mathbf{X})$.

A *canonical scale* on a metric group (G, d) is a map $\mathcal{S}_G : G \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by

$$\mathcal{S}_G(g, r) = \max \left\{ r, \sup \{ d(g^{-1} h g, e) : h \in G, d(h, e) \leq r \} \right\}.$$

Let $\mathbf{X} = (\overline{X}, e, d, \Gamma)$ be a scaled space, (G, d_G) be a metric group and Γ_G be a scale on G . A map $\phi : \overline{X} \rightarrow G$ is called a *Lipschitz morphism with respect to the scale Γ_G* if for all $x, y \in \overline{X}$ and all $r \in \mathbb{R}^+$

- (i) $\phi(e) = e$;
- (ii) $\phi(x^{-1}) = \phi(x)^{-1}$;
- (iii) $d_G(\phi(x), \phi(y)) \leq d(x, y)$;
- (iv) $\Gamma_G(\phi(x), r) \leq \Gamma(x, r)$.

We say that ϕ is a Lipschitz morphism, if it is a Lipschitz morphism with respect to the canonical scale \mathcal{S}_G .

For a scaled space \mathbf{X} let $\overline{F}(\mathbf{X})$ denote the group completion of $F(\mathbf{X})$. Note that $\overline{F}(\mathbf{X})$ is usually not a free group.

Proposition 1.3 (Lemma 3.7, [4]). *Let ϕ be a Lipschitz morphism from a scaled space \mathbf{X} into a metric group G . The map ϕ extends to a Lipschitz homomorphism $\phi : F(\mathbf{X}) \rightarrow G$. If G is completely metrizable, then ϕ can be further extended to a continuous homomorphism $\phi : \overline{F}(\mathbf{X}) \rightarrow G$.*

2. UNIONS OF SCALED SPACES

Definition 2.1. Given two scaled spaces $\mathbf{X} = (\overline{X}, e, d_X, \Gamma_X)$ and $\mathbf{Y} = (\overline{Y}, e, d_Y, \Gamma_Y)$ we define their union $\mathbf{Z} = (\overline{Z}, e, d, \Gamma)$ to be the amalgam of \overline{X} and \overline{Y} over e . More precisely, if Z is the amalgam of metric spaces X and Y over the subspace e , then as a metric space $\mathbf{X} \cup \mathbf{Y}$ is obtained from Z by adding formal inverses. Note that \overline{Z} is also the amalgam of \overline{X} and \overline{Y} over e ; in other words $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$. The scale Γ on \overline{Z} is the union of scales Γ_X and Γ_Y :

$$\Gamma(z, r) = \begin{cases} \Gamma_X(z, r) & \text{if } z \in \overline{X}, \\ \Gamma_Y(z, r) & \text{if } z \in \overline{Y}. \end{cases}$$

The union of scaled spaces \mathbf{X} and \mathbf{Y} is denoted by $\mathbf{X} \cup \mathbf{Y}$.

Let \mathbf{X} and \mathbf{Y} be scaled spaces and let $\pi_X : \overline{X \cup Y} \rightarrow \overline{X}$ be the retract map

$$\pi_X(z) = \begin{cases} z & \text{if } z \in \overline{X}, \\ e & \text{if } z \in \overline{Y}. \end{cases}$$

This map is Lipschitz and it extends to a surjective group homomorphism $\pi_X : F(\mathbf{X} \cup \mathbf{Y}) \rightarrow F(\mathbf{X})$.

Proposition 2.2. *The homomorphism $\pi_X : F(\mathbf{X} \cup \mathbf{Y}) \rightarrow F(\mathbf{X})$ is Lipschitz.*

Proof. Let $f \in F(\mathbf{X} \cup \mathbf{Y})$ be given. Let N_X be the Graev norm on $F(\mathbf{X})$ and let N be the Graev norm on $F(\mathbf{X} \cup \mathbf{Y})$. Pick $w \in W(\overline{X \cup Y})$, $\hat{w} = f$, and a match θ on w . We need to show that $N_X(\pi_X(f)) \leq N(f)$, and for this it is enough to find a word $u \in W(\overline{X})$ and a match μ on u such that $\hat{u} = \pi_X(f)$ and $N^\mu(u) \leq N^\theta(w)$ (note that $N_X^\mu(u) = N^\mu(u)$ because \mathbf{X} is a subspace of $\mathbf{X} \cup \mathbf{Y}$, and therefore we omit the subscript). If $w = z_1 \cdots z_n$ with $z_i \in \overline{X \cup Y}$, set $u = \tilde{z}_1 \cdots \tilde{z}_n$ with $\tilde{z}_i = \pi_X(z_i)$. We can view θ as being also a match on u . Since $\pi_X : \overline{X \cup Y} \rightarrow \overline{X}$ is Lipschitz, we have $d(z_i, z_j^{-1}) \geq d(\tilde{z}_i, \tilde{z}_j^{-1})$ for all i, j . By item (i) of the definition of the scale, $\Gamma(z, r) \geq \Gamma(\pi_X(z), r)$ for all $r \in \mathbb{R}^+$ and all $z \in \overline{X \cup Y}$. It now follows from item (iii) of the scale and from the definition of the norm that $N^\theta(u) \leq N^\theta(w)$. \square

The homomorphism π_X being Lipschitz is therefore continuous and thus extends to a continuous homomorphism $\pi_X : \overline{F}(\mathbf{X} \cup \mathbf{Y}) \rightarrow \overline{F}(\mathbf{X})$. Note that $\pi_X(f) = f$ for any $f \in \overline{F}(\mathbf{X})$ and that $\pi_X(f) = e$ holds for all $f \in \overline{F}(\mathbf{Y})$. Of course, we also have a continuous homomorphism $\pi_Y : \overline{F}(\mathbf{X} \cup \mathbf{Y}) \rightarrow \overline{F}(\mathbf{Y})$.

Corollary 2.3. *Let \mathbf{X} and \mathbf{Y} be scaled spaces and let $\mathbf{X} \cup \mathbf{Y}$ be their union. The inclusion $F(\mathbf{X}) \hookrightarrow F(\mathbf{X} \cup \mathbf{Y})$ is isometric.*

Proof. Let N denote the Graev norm on $F(\mathbf{X} \cup \mathbf{Y})$ and let N_X be the Graev norm on $F(\mathbf{X})$. We need to show that for all $f \in F(\mathbf{X})$ one has $N(f) = N_X(f)$. By definition

$$\begin{aligned} N_X(f) &= \inf \left\{ N^\theta(w) : w \in W(\overline{X}), \hat{w} = f \text{ and } \theta \text{ is a match on } w \right\}, \\ N(f) &= \inf \left\{ N^\theta(w) : w \in W(\overline{X \cup Y}), \hat{w} = f \text{ and } \theta \text{ is a match on } w \right\}, \end{aligned}$$

and therefore $N(f) \leq N_X(f)$. The reverse inequality follows immediately from $\pi_X(f) = f$ for $f \in F(\mathbf{X})$ and Proposition 2.2. \square

Corollary 2.4. *Inclusion $F(\mathbf{X}) \hookrightarrow F(\mathbf{X} \cup \mathbf{Y})$ extends to $\overline{F}(\mathbf{X}) \hookrightarrow \overline{F}(\mathbf{X} \cup \mathbf{Y})$.*

3. FREE PRODUCTS OF POLISH GROUPS

For this section we fix two Polish groups G and H . Scaled spaces are assumed to be separable, and therefore the groups $\overline{F}(\mathbf{X})$ are Polish.

Let \mathbf{X} and \mathbf{Y} be scaled spaces, and let $\phi_G : \mathbf{X} \rightarrow G$ and $\phi_H : \mathbf{Y} \rightarrow H$ be *surjective* Lipschitz morphisms with respect to some compatible left-invariant metrics on G and H . By Proposition 1.3 they extend to surjective homomorphisms $\phi_G : \overline{F}(\mathbf{X}) \rightarrow G$ and $\phi_H : \overline{F}(\mathbf{Y}) \rightarrow H$ with kernels \mathfrak{N}_G and \mathfrak{N}_H respectively. We note that as proved in [4, Theorem 3.10], for any Polish group G there are plenty of surjective Lipschitz morphisms $\phi : \overline{X} \rightarrow G$, and moreover, one may always take $X = \mathbb{N}^{\mathbb{N}}$. Note also that G is isomorphic to $\overline{F}(\mathbf{X})/\mathfrak{N}_G$ with the quotient topology (see, for instance, [1, Theorem 1.2.6]). We shall identify G with $\overline{F}(\mathbf{X})/\mathfrak{N}_G$ and H with $\overline{F}(\mathbf{Y})/\mathfrak{N}_H$. By Corollary 2.4 we can view \mathfrak{N}_G and \mathfrak{N}_H as subgroups of $\overline{F}(\mathbf{X} \cup \mathbf{Y})$. Let \mathfrak{N}_{G*H}° be the normal subgroup of $\overline{F}(\mathbf{X} \cup \mathbf{Y})$ generated by \mathfrak{N}_G and \mathfrak{N}_H

$$\mathfrak{N}_{G*H}^\circ = \left\{ f_1 w_1 f_1^{-1} \cdots f_n w_n f_n^{-1}, \quad n \in \mathbb{N}, f_i \in \overline{F}(\mathbf{X} \cup \mathbf{Y}), w_i \in \langle \mathfrak{N}_G, \mathfrak{N}_H \rangle \right\}.$$

Let \mathfrak{N}_{G*H} be the closure of \mathfrak{N}_{G*H}° .

Lemma 3.1. *In the setting above $\pi_X(\mathfrak{N}_{G*H}) = \mathfrak{N}_G$.*

Proof. If $w \in \mathfrak{N}_{G*H}^\circ$ is of the form

$$w = f_1 w_1 f_1^{-1} \cdots f_n w_n f_n^{-1}, \quad f_i \in \overline{F}(\mathbf{X} \cup \mathbf{Y}), w_i \in \langle \mathfrak{N}_G, \mathfrak{N}_H \rangle,$$

then

$$\pi_X(w) = \pi_X(f_1) \pi_X(w_1) \pi_X(f_1)^{-1} \cdots \pi_X(f_n) \pi_X(w_n) \pi_X(f_n)^{-1} \in \mathfrak{N}_G,$$

because $\pi_X(w_i) \in \mathfrak{N}_G$, $\pi_X(f_i) \in \overline{F}(\mathbf{X})$ and \mathfrak{N}_G is a normal subgroup in $\overline{F}(\mathbf{X})$. Thus $\pi_X(\mathfrak{N}_{G*H}^\circ) = \mathfrak{N}_G$ and therefore also $\pi_X(\mathfrak{N}_{G*H}) = \mathfrak{N}_G$, since \mathfrak{N}_G is closed. \square

A Polish free product of G and H over ϕ_G and ϕ_H is the group $\overline{F}(\mathbf{X} \cup \mathbf{Y})/\mathfrak{N}_{G*H}$; we denote it by $G \underset{\phi_G * \phi_H}{\ast} H$. The free product comes with homomorphisms $\iota_G : G \rightarrow G \underset{\phi_G * \phi_H}{\ast} H$, $\iota_H : H \rightarrow G \underset{\phi_G * \phi_H}{\ast} H$ and $\pi_G : G \underset{\phi_G * \phi_H}{\ast} H \rightarrow G$, $\pi_H : G \underset{\phi_G * \phi_H}{\ast} H \rightarrow H$ given by

$$\begin{aligned} \iota_G(f \mathfrak{N}_G) &= f \mathfrak{N}_{G*H}, & \iota_H(f \mathfrak{N}_H) &= f \mathfrak{N}_{G*H}, \\ \pi_G(f \mathfrak{N}_{G*H}) &= \pi_X(f) \mathfrak{N}_G, & \pi_H(f \mathfrak{N}_{G*H}) &= \pi_Y(f) \mathfrak{N}_H. \end{aligned}$$

Note that π_G and π_H are well-defined by Lemma 3.1. Note also that $\pi_G(\iota_G(g)) = g$ and $\pi_H(\iota_H(h)) = h$ for all $g \in G$ and $h \in H$.

Proposition 3.2. *Let \mathbf{X} , \mathbf{Y} , ϕ_G , ϕ_H and $G \underset{\phi_G * \phi_H}{\ast} H$ be as above. Let d_G and d_H be compatible left-invariant metrics on G and H with respect to which ϕ_G and ϕ_H are Lipschitz morphisms.*

- (i) *The homomorphisms ι_G and ι_H are injective.*
- (ii) *The homomorphisms ι_G and ι_H are continuous.*
- (iii) *The homomorphisms π_G and π_H are continuous.*
- (iv) *$\iota_G(G) \cap \iota_H(H) = \{e\}$.*
- (v) *The group generated by $\iota_G(G)$ and $\iota_H(H)$ is a dense subgroup of $G \underset{\phi_G * \phi_H}{\ast} H$.*
- (vi) *If (T, d_T) is a Polish metric group, $\psi_G : G \rightarrow T$, $\psi_H : H \rightarrow T$ are Lipschitz homomorphisms (with respect to d_G and d_H) and $\mathcal{S}_T(\psi_G(g), r) \leq \mathcal{S}_G(g, r)$, $\mathcal{S}_T(\psi_H(h), r) \leq \mathcal{S}_H(h, r)$ for all $g \in G$, $h \in H$ and $r \in \mathbb{R}^+$, then there is a unique continuous homomorphism $\psi : G \underset{\phi_G * \phi_H}{\ast} H \rightarrow T$ such that $\psi \circ \iota_G = \psi_G$ and $\psi \circ \iota_H = \psi_H$.*

Proof. (i) To show that ι_G is injective it is enough to check that $\mathfrak{N}_{G*H} \cap \overline{F}(\mathbf{X}) = \mathfrak{N}_G$. If $f \in \mathfrak{N}_{G*H} \cap \overline{F}(\mathbf{X})$, then $f = \pi_X(f) \in \pi_X(\mathfrak{N}_{G*H}) = \mathfrak{N}_G$, by Lemma 3.1; hence $f \in \mathfrak{N}_G$.

(ii) Let d be a compatible right-invariant metric on $\overline{F}(\mathbf{X} \cup \mathbf{Y})$. This metric induces compatible right-invariant metrics on the factor groups $\overline{F}(\mathbf{X} \cup \mathbf{Y})/\mathfrak{N}_{G*H}$ and $\overline{F}(\mathbf{X})/\mathfrak{N}_G$ (see [5, Lemma 2.2.8])

$$\begin{aligned} d_1(f_1 \mathfrak{N}_{G*H}, f_2 \mathfrak{N}_{G*H}) &= \inf \{ d(f_1 k_1, f_2 k_2) : k_1, k_2 \in \mathfrak{N}_{G*H} \}, \\ d_2(f_1 \mathfrak{N}_G, f_2 \mathfrak{N}_G) &= \inf \{ d(f_1 k_1, f_2 k_2) : k_1, k_2 \in \mathfrak{N}_G \}. \end{aligned}$$

With respect to the metrics d_1 and d_2 the homomorphism ι_G is Lipschitz, hence continuous.

(iii) It is enough to prove that π_G is continuous at the identity, i.e., that $f_n \mathfrak{N}_{G*H} \rightarrow \mathfrak{N}_{G*H}$ implies $\pi_X(f_n) \mathfrak{N}_G \rightarrow \mathfrak{N}_G$. The sequence $f_n \mathfrak{N}_{G*H}$ converges to \mathfrak{N}_{G*H} if and only if there is a sequence of $h_n \in \mathfrak{N}_{G*H}$ such that $f_n h_n \rightarrow e$. This implies $\pi_X(f_n) \pi_X(h_n) \rightarrow e$ with $\pi_X(h_n) \in \mathfrak{N}_G$ by Lemma 3.1, and therefore $\pi_G(f_n \mathfrak{N}_{G*H}) = \pi_X(f_n) \mathfrak{N}_G \rightarrow \mathfrak{N}_G$.

(iv) If $f \mathfrak{N}_{G*H} \in \iota_G(G) \cap \iota_H(H)$, then $f \mathfrak{N}_{G*H} = f_1 \mathfrak{N}_{G*H} = f_2 \mathfrak{N}_{G*H}$ for some $f_1 \in \overline{F}(\mathbf{X})$ and $f_2 \in \overline{F}(\mathbf{Y})$. Therefore $\pi_X(f \mathfrak{N}_{G*H}) = \pi_X(f_1 \mathfrak{N}_{G*H}) = \pi_X(f_2 \mathfrak{N}_{G*H})$, but $\pi_X(f_1 \mathfrak{N}_{G*H}) = f_1 \mathfrak{N}_G$ and $\pi_X(f_2 \mathfrak{N}_{G*H}) = \mathfrak{N}_G$, whereby $f_1 \in \mathfrak{N}_G$ and thus $f \mathfrak{N}_{G*H} = \mathfrak{N}_{G*H}$.

(v) This item is obvious, since the group generated by the images of ι_G and ι_H is nothing else but $\langle \overline{F}(\mathbf{X}), \overline{F}(\mathbf{Y}) \rangle \mathfrak{N}_{G*H}$.

(vi) Maps $\psi_G \circ \phi_G : \overline{X} \rightarrow T$ and $\psi_H \circ \phi_H : \overline{Y} \rightarrow T$ are Lipschitz morphisms and so is the map $\xi : \overline{X \cup Y} \rightarrow T$ given by

$$\xi(z) = \begin{cases} \psi_G \circ \phi_G(z) & \text{if } z \in \overline{X}, \\ \psi_H \circ \phi_H(z) & \text{if } z \in \overline{Y}. \end{cases}$$

By Proposition 1.3 ξ extends to a continuous homomorphism $\xi : \overline{F}(\mathbf{X} \cup \mathbf{Y}) \rightarrow T$. Since ξ extends both $\psi_G \circ \phi_G$ and $\psi_H \circ \phi_H$, the kernel of ξ contains \mathfrak{N}_G and \mathfrak{N}_H , and therefore also \mathfrak{N}_{G*H} . Thus ξ factors to a continuous homomorphism $\psi : G \underset{\phi_G * \phi_H}{*} H \rightarrow T$. Uniqueness follows from item (v). \square

Note that items (i), (ii) and (iii) imply that ι_G and ι_H are embeddings. The homomorphisms ι_G, ι_H can be extended to a homomorphism from the free product of abstract groups $\iota : G * H \rightarrow G \underset{\phi_G * \phi_H}{*} H$.

Question 3.3. *Is the homomorphism ι injective for all choices of ϕ_G, ϕ_H ?*

While we cannot answer this question in full generality, we shall show in Corollary 3.6 that the answer is yes when ϕ_G and ϕ_H are “large enough”.

Lemma 3.4. *Let $f \in G * H$ be a non-trivial element in the abstract free product. There are a Polish group T and two embeddings $\psi_G : G \rightarrow T$ and $\psi_H : H \rightarrow T$ such that for the common extension of these homomorphisms $\psi : G * H \rightarrow T$ one has $\psi(f) \neq e$.*

Proof. Let $f \in G * H$ be given. By conjugating f with an appropriate $\beta \in G * H$ we may assume without loss of generality that f is of the form $f = g_{n-1} h_{n-1} \cdots g_0 h_0$ for some $n \geq 1$ with non-trivial $g_i \in G$ and $h_i \in H$. By a theorem of Uspenskij [7] (see also [6, Theorem 9.18]) the group $\text{Homeo}([0, 1]^\omega)$ of homeomorphisms of the Hilbert Cube with the compact-open topology is a universal Polish group in the following sense: any Polish group can be embedded into $\text{Homeo}([0, 1]^\omega)$. In particular, G and H can be embedded into $\text{Homeo}([0, 1]^\omega)$; to simplify notations we assume that G and H are actual subgroups of $\text{Homeo}([0, 1]^\omega)$. Note that $\alpha H \alpha^{-1}$ is a copy of H inside $\text{Homeo}([0, 1]^\omega)$ for any $\alpha \in \text{Homeo}([0, 1]^\omega)$. To prove the lemma it is therefore sufficient to construct a homeomorphism $\alpha \in \text{Homeo}([0, 1]^\omega)$ such that for some $x_0 \in [0, 1]^\omega$

$$g_{n-1} \alpha h_{n-1} \alpha^{-1} \cdots g_0 \alpha h_0 \alpha^{-1}(x_0) \neq x_0.$$

Pick any $x_0 \in [0, 1]^\omega$ and any $x_1 \in [0, 1]^\omega$ such that $x_1 \neq x_0$ and $h_0(x_1) \notin \{x_1, x_0\}$; set $x_2 = h_0(x_1)$. Pick any $x_3 \in \text{Homeo}([0, 1]^\omega)$ such that $x_3 \notin \{x_0, x_1, x_2\}$ and $g_0(x_3) \notin \{x_0, x_1, x_2, x_3\}$; set $x_4 = g_0(x_3)$. Note that if $h \in \text{Homeo}([0, 1]^\omega)$ is non-trivial, there is an open set $U \subseteq [0, 1]^\omega$ such that $h(x) \neq x$ for all $x \in U$. Using this observation and that all g_i and h_i are non-trivial we continue in this fashion and construct a sequence $(x_k)_{k=1}^{4n}$ such that

- (i) $x_i \neq x_j$ for $i \neq j$;
- (ii) $h_k(x_{4k+1}) = x_{4k+2}$ for $k = 0, \dots, n-1$;
- (iii) $g_k(x_{4k+3}) = x_{4k+4}$ for $k = 0, \dots, n-1$.

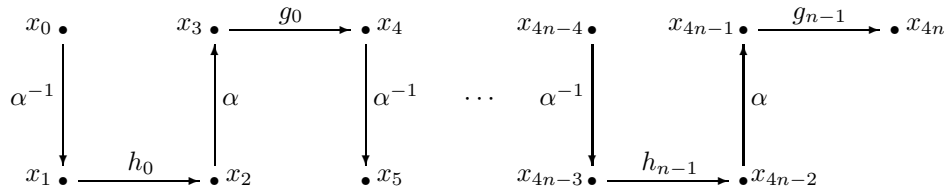


FIGURE 1. Construction of the homeomorphism α .

For all $m \in \mathbb{N}$ the space $[0, 1]^\omega$ is m -homogeneous: for tuples (y_1, \dots, y_m) and (z_1, \dots, z_m) of distinct elements there is a homeomorphism $h \in \text{Homeo}([0, 1]^\omega)$ such that $h(y_i) = z_i$ (see, for example, [8, Exercise 2, p. 261]). Whence there is some $\alpha \in \text{Homeo}([0, 1]^\omega)$ such that $\alpha(x_{4k+1}) = x_{4k}$ and $\alpha(x_{4k+2}) = x_{4k+3}$ for all $k = 0, \dots, n-1$. For such an α we have

$$g_{n-1}\alpha h_{n-1}\alpha^{-1} \cdots g_0\alpha h_0\alpha^{-1}(x_0) = x_{4n},$$

and $x_{4n} \neq x_0$ by construction. \square

Theorem 3.5. *There are a Polish group T and embeddings $\psi_G : G \hookrightarrow T$, $\psi_H : H \hookrightarrow T$ such that the group $\langle \psi_G(G), \psi_H(H) \rangle$ is naturally isomorphic to the group $G * H$.*

Proof. Lemma 3.4 implies that for any non-trivial $f \in G * H$ we may fix a Polish group T_f and a homomorphism $\psi_f : G * H \rightarrow T_f$ such that $\psi_f|_G : G \rightarrow T_f$ and $\psi_f|_H : H \rightarrow T_f$ are embeddings and $\psi_f(f) \neq e$.

Let $f \in G * H$ be given and assume for notational simplicity that f has form $g_{n-1}h_{n-1} \cdots g_0h_0$ for some non-trivial $g_i \in G$, $h_i \in H$ and $n \geq 1$. By continuity of $\psi_f|_G$, $\psi_f|_H$ and since $\psi_f(f) \neq e$, there are neighborhoods $U_i^{(f)} \subseteq G$ of g_i and $V_i^{(f)} \subseteq H$ of h_i such that $e \notin U_{n-1}^{(f)}V_{n-1}^{(f)} \cdots U_0^{(f)}V_0^{(f)}$. Therefore we can select a countable family $(f_m)_{m=1}^\infty$ of elements $f_m \in G * H$ such that for any $f \in G * H$ there is some m with $\psi_{f_m}(f) \neq e$.

Let $T = \prod_m T_{f_m}$ be the direct product of the groups T_{f_m} and let $\psi : G * H$ be the homomorphism given by $\psi(f)(m) = \psi_{f_m}(f)$. The homomorphisms $\psi|_G : G \rightarrow T$ and $\psi|_H : H \rightarrow T$ are embeddings. By the choice of the family (f_m) we also have $\psi(f) \neq e$ for any non-trivial $f \in G * H$ and therefore the homomorphism ψ is injective. \square

Corollary 3.6. *There are left-invariant compatible metrics d_G and d_H on G and H respectively and scales Γ_G and Γ_H on (G, d_G) and (H, d_H) with the following property: if \mathbf{X} and \mathbf{Y} are scaled spaces and $\phi_G : \mathbf{X} \rightarrow G$, $\phi_H : \mathbf{Y} \rightarrow H$ are surjective Lipschitz morphisms with respect to the scales Γ_G and Γ_H , then the canonical homomorphism $\iota : G * H \rightarrow G_{\phi_G * \phi_H} H$ is injective.*

Proof. By Theorem 3.5 we may assume that G and H are closed subgroups of a Polish group T and that $\langle G, H \rangle$ is canonically isomorphic to $G * H$. Let d be a compatible left-invariant metric on T and let d_G and d_H be the restrictions of d onto G and H respectively. Finally, let Γ_G and Γ_H be the restrictions of \mathcal{S}_T onto G and H . By item (vi) of Proposition 3.2 the maps ϕ_G and ϕ_H extend to a homomorphism $\phi : G_{\phi_G * \phi_H} H \rightarrow T$. Since $\langle G, H \rangle \cong G * H$, the homomorphism $\iota : G * H \rightarrow G_{\phi_G * \phi_H} H$ must be injective. \square

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